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Note on the Busy Period in the Case of Infinite Means

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We consider in this note an ordinary single server queue in which the service time of the first customer is an arbitrary constant b, the service times of succeeding customers are independent, identically distributed random variables with an arbitrary distribution function G, and the inter-arrival times are independent, identically distributed random variables with an arbitrary distribution function F. (We take these distribution functions to be continuous from the right). Assume that there is no point x_o such that $G(x_o) - G(x_o-o) = F(x_o) - F(x_o-o) = 1$ (non-degeneracy assumption). Put

$$\mu_G = \int_{[o,\infty)} x dG(x),$$

and

$$\mu_F = \int_{[o,\infty)} x dF(x).$$

It is known [1] that if $\mu_G < \infty$ and $\mu_G \leq \mu_F$, then the probability, p(b), that the busy period never terminates is 0. If $\mu_F < \mu_G \leq \infty$, then $p(b) \longrightarrow 1$ as $b \longrightarrow \infty$. When both distributions degenerate at x_o, $p(b) = 1$ for all $b \geq x_o$. The method used to derive these results breaks down when $\mu_F = \mu_G = \infty$. This case, which is a little more delicate, is the subject of this note.

Put $T(x) = 1 - G(x)$, $R(x) = 1 - F(x)$, $\tilde{T}(s) = \int_{[o,\infty)} e^{-sx} T(x) dx$, $\tilde{R}(s) = \int_{[o,\infty)} e^{-sx} R(x) dx$, and

$$Q(s) = \frac{\tilde{R}(s)}{\tilde{T}(s)}, \quad s > o. \tag{1}$$

If at least one of the means (μ_F or μ_G) is finite, $Q(0)$ ($= \frac{\mu_F}{\mu_G}$) is well-defined, though possibly infinite, and determines the character of $p(b)$. When $\mu_F = \mu_G = \infty$, $Q(0)$ evidently is undefined, but we might anticipate that the behavior of $Q(s)$ in a neighborhood of $s = 0$ is the governing quantity. That this is indeed the case is shown, in Theorem 1, for the proof of which we need a lemma.

LEMMA. Let $A(t)$, $B(t)$, $0 \leq t < \infty$, be functions such that

(i) $$B(t) \geq 0,$$

(ii) $$\int_{[0,\infty)} B(t)dt = \infty,$$

and

(iii) $$\lim_{t \to \infty} \frac{A(t)}{B(t)} = \lambda.$$

Then

$$\lim_{s \downarrow 0} \frac{\int_{[0,\infty)} e^{-st} A(t)dt}{\int_{[0,\infty)} e^{-st} B(t)dt} = \lambda.$$

Proof: Assume $\lambda = 0$, let $\varepsilon > 0$ be arbitrary, and choose t_0 such that $|A(t)| < \varepsilon\, B(t)$ for $t \geq t_0$. Then,

$$\limsup_{s \downarrow 0} \frac{\left| \int_{[0,\infty)} e^{-st} A(t)\,dt \right|}{\int_{[0,\infty)} e^{-st} B(t)\,dt}$$

$$\leq \limsup_{s \downarrow 0} \frac{\int_{[0,t_0)} |A(t)|\,dt + \varepsilon \int_{[t_0,\infty)} e^{-st} B(t)\,dt}{\int_{[0,\infty)} e^{-st} B(t)\,dt} \leq \varepsilon.$$

Since ε is arbitrary, the proof is complete for $\lambda = 0$. For general λ we need only replace $A(t)$ by $A(t) - \lambda B(t)$.

THEOREM 1. If $\limsup_{s \downarrow 0} Q(s) > 0$, then $p(b) = 0$ for all $b \geq 0$.

Proof: We have

$$\tilde{F}(s) \equiv \int_{[0,\infty)} e^{-sx}\,dF(x) = 1 - s\tilde{R}(s)$$

and

$$\tilde{G}(s) \equiv \int_{[0,\infty)} e^{-sx}\,dG(x) = 1 - s\tilde{T}(s).$$

It is shown in [1] that $p(b)$ satisfies the functional equation

(2) $$p(b) = \int_{[0,\infty)} \int_{[0,b]} p(b-t+x)\,dF(t)\,dG(x),$$

and the transforms satisfy the equation

(3) $$\tilde{p}(s) \equiv \int_{[0,\infty)} e^{-sx} p(x)\,dx = \tilde{F}(s) \int_{[0,\infty)} \phi_s(x)\,dG(x),$$

where

$$\phi_s(x) = \begin{cases} \int_{[0,\infty)} e^{-st} p(t+x)\,dt & \text{if} \quad x \geq 0 \\ \phi_s(0) & \text{otherwise.} \end{cases}$$

Since

$$\tilde{p}(s) = \int_{[0,\infty)} \int_{[0,\infty)} e^{-sx} p(x)\,dx\,dG(t) = \int_{[0,\infty)} \int_{[0,t)} e^{-sx} p(x)\,dx\,dG(t)$$

$$+ \int_{[0,\infty)} \int_{[t,\infty)} e^{-sx} p(x)\,dx\,dG(t),$$

we obtain (by interchanging the order of integration in the first term on the right-hand side and changing variables in the second term)

$$\tilde{p}(s) = \int_{[0,\infty)} e^{-sx} p(x) T(x)\,dx + \int_{[0,\infty)} e^{-st} \phi_s(t)\,dG(t).$$

Subtracting $\tilde{G}(s) \int_{[0,\infty)} \phi_s(x)\,d\,G(x)$ from both sides of (3) yields

$$(4) \quad \int_{[0,\infty)} e^{-sx} p(x) T(x)\,dx + \int_{[0,\infty)} \phi_s(x) \left(e^{-sx} - \tilde{G}(s)\right) dG(x)$$

$$= s\left(\tilde{T}(s) - \tilde{R}(s)\right) \int_{[0,\infty)} \phi_s(x)\,dG(x).$$

It is shown in [1] that $p(b)$ is non-decreasing in b. Put

$$p(\infty) = \lim_{b \to \infty} p(b).$$

Then rewriting equation (4) in the more convenient form

$$(5)\quad \frac{\int_{[0,\infty)} e^{-sx}A(x)dx}{\int_{[0,\infty)} e^{-sx}B(x)dx} - \left(1-\tilde{Q}(s)\right)\int_{[0,\infty)} s\Phi_s(x)dG(x)$$

$$= - \frac{\int_{[0,\infty)} \Phi_s(x) \left(e^{-sx}-\tilde{G}(s)\right)dG(x)}{\tilde{T}(s)}$$

where $A(x) = p(x)T(x)$ and $B(x) = T(x)$, we obtain by lemma 1, bounded convergence, and the Abelian theorem for Laplace transforms

$$(6)\quad p(\infty) \lim_{s \downarrow 0} \sup Q(s) = - \lim_{s \downarrow 0} \sup \frac{\int_{[0,\infty)} \Phi_s(x) \left(e^{-sx}-\tilde{G}(s)\right)dG(x)}{\tilde{T}(s)}.$$

We shall now show that the right-hand side of (6) is zero. Putting

$$\psi_s(x) = \Phi_s(x) - \frac{p(\infty)}{s},$$

and letting $\varepsilon > 0$ be arbitrary, we note that

$$\frac{\int_{[0,\infty)} \Phi_s(x) \left(e^{-sx}-\tilde{G}(s)\right) dG(x)}{\tilde{T}(s)} = - \frac{\int_{[0,x_0)} (1-e^{-sx})\psi_s(x)dG(x)}{\tilde{T}(s)}$$

$$- \frac{\int_{[x_0,\infty)} (1-e^{-sx})\psi_s(x)dG(x)}{\tilde{T}(s)} + \int_{[0,\infty)} s\,\psi_s(x)dG(x),$$

where x_o is chosen so that $p(\infty) - p(x) < \varepsilon$ for $x \geq x_o$. By bounded convergence and the Abelian theorem for Laplace transforms,

$$\lim_{s \downarrow 0} \int_{[0,\infty)} s\, \psi_s(x) dG(x) = 0.$$

Moreover, since $|s\Phi_s(x) - p(\infty)| < \varepsilon$ for $x \geq x_o$, $|s\, \psi_s(x)| \leq 1$, and $1-e^{-sx} \leq sx$, we have

$$\frac{\left| \int_{[x_o,\infty)} (1-e^{-sx})\psi_s(x) dG(x) \right|}{\tilde{T}(s)} \leq \varepsilon,$$

and

$$\frac{\left| \int_{[0,x_o)} (1-e^{-sx})\psi_s(x) dG(x) \right|}{\tilde{T}(s)} \leq \frac{\int_{[0,x_o)} x dG(x)}{\tilde{T}(s)} \downarrow 0 \text{ as } s \downarrow 0.$$

Thus

$$\lim_{s \downarrow 0} \frac{\int_{[0,\infty)} \Phi_s(x)(e^{-sx} - \tilde{G}(s)) dG(x)}{\tilde{T}(s)} = 0,$$

which, by (6), implies $p(\infty) = 0$. This completes the proof.

By methods given in [1] the above result can be shown to hold for another queuing process closely related to the conventional one; namely, the moving queue with an absorbing barrier. In this process an assembly line moving toward a point 0 with uniform speed, has items spaced for service along it. If an

item arrives at 0 before service on it has been completed, the server suffers a probability α of being disabled (absorbed). In the terminology of the moving queue, theorem 1 has the following formulation:

THEOREM 2. Let $G(x)$ be the distribution function of the distance between adjacent elements and $F(x)$ the service time distribution. Let $p(b)$ denote the probability of serving infinitely many members of the assembly line, if service on the first elements begins when it is b units away from the barrier. If $\alpha > 0$, $\mu_F = \mu_G = \infty$, and $\limsup_{s \downarrow 0} Q(s) > 0$, then $p(b) = 0$ for all $b \geq 0$.

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